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HIERARCHIES OF REGIONAL SUB-STRUCTURES AND THEIR MULTIPLIERS WITHIN INPUT-OUTPUT SYSTEMS: MIYAZAWA REVISITED

MICHAEL SONIS AND GEOFFREY J.D. HEWINGS*

Abstract

Miyazawa, in a series of papers, provided some new insights into the functioning of an economy by revealing, explicity, the important linkages between output and income propagation effects. The concept of an interrelational income multiplier has provided the basis for extensions of this work in the field of demographic-economic modeling. In the present paper, some of Miyazawa's contributions are revisited and some extensions proposed that exploit some new applications of matrix decomposition techniques. In particular, a finer structure for Miyazawa's concepts of "internal" and "external" multipliers are revealed. The final section of the paper explores the decomposition approach to the Miyazawa demo-economic framework in a multi-regional setting.

I. *Introduction*

The contributions of Ken'ichi Miyazawa (1960, 1966, 1968, 1971, 1976) have had an important impact on the development of extended input-output analysis; in particular, his work has stimulated some new thinking about the design and implementation of multiregional economic systems. Perhaps, the most important contribution was in the formulation of interrelational income multipliers and his alternative procedures for decomposing an economic system to reveal the contributions of income change separately from output change. In regional analysis, Richardson (1972) noted the importance of the matrix of income multipliers while Miyazawa's partition methods were evident in the extended demoeconomic models [for a summary, see Batey (1985), Dewhurst, Hewings and Jensen (1991) and Sonis and Hewings (1991)].

The purpose of this short paper is to clarify the Miyazawa contributions from the viewpoint of synergetic interactions of regional sub-systems and to reveal a connection between the Miyazawa analytical formalism and the work of the Frobenius school that dates from the beginning of this century. The paper also provides some further directions for the elaboration of partitioned input-output analysis.

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II. *Multiplicative Structure of the Leontief Inverse and the Miyazawa Partitioned Matrix Multiplier*

Consider a two-region input-output system represented by the following block matrix, A, of direct inputs:

$$A=\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \tag{1}$$

where A_{11} and A_{22} are the quadrat matrices of direct inputs within the first and second region and A_{12} and A_{21} are the rectangular matrices showing the direct inputs purchased by the second region and vice versa. The matrix, A, can be presented in a separate form, which will be referred to as a "pull-decomposition." In this perspective, the first region is shown to exert an influence on the second region by pulling inputs (i.e., imports) for production from this second region. A similar perspective applies to the second region's interaction with the first region. Hence, depending upon the perspective employed, the off diagonal entries of (1) may be viewed as "push" or "pull" linkages with the other region.

$$A=\left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & 0 \end{array}\right)+\left(\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & A_{22} \end{array}\right)=A_1+A_2 \tag{2}$$

If the Leontief inverse exists for the first region, it will be defined as follows:

$$B_1=(I-A_{11})^{-1} \tag{3}$$

and following Miyazawa, this will be referred to as the internal matrix multiplier for the first region.

Consider the block-matrix:

$$G_1=(I-A_1)^{-1} \tag{4}$$

and, from direct matrix multiplication, the following will be obtained:

$$G_1=\left(\begin{array}{c|c} B_1 & 0 \\ \hline A_{21}B_1 & I \end{array}\right)=\left(\begin{array}{c|c} I & 0 \\ \hline A_{21} & I \end{array}\right)\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & I \end{array}\right) \tag{5}$$

Further:

$$G_1(I-A)=G_1[(I-A_1)-A_2]=I-G_1A_2 \tag{6}$$

or:

$$I-G_1A_2=\left(\begin{array}{c|c} I & -B_1A_{12} \\ \hline 0 & I-A_{22}-A_{21}B_1A_{12} \end{array}\right) \tag{7}$$

The Leontief inverse may be defined as:

$$\Delta_2=(I-A_{22}-A_{21}B_1A_{12})^{-1} \tag{8}$$

and this is referred to as the external matrix multiplier of the second region revealing the influence of inputs from the first region.[3]

Furthermore, consider the block-matrix:

$$G_2=(I-G_1A_2)^{-1} \tag{9}$$

from which, direct matrix multiplication implies that:

$$G_2=\left(\begin{array}{c|c} I & B_1A_{12}\Delta_2 \\ \hline 0 & \Delta_2 \end{array}\right)=\left(\begin{array}{c|c} I & B_1A_{12} \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & \Delta_2 \end{array}\right) \tag{10}$$

Moreover,

$$G_2G_1(I-A)=I$$

or

$$(I-A)^{-1}=G_2G_1=\left(\begin{array}{c|c} I & B_1A_{12} \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & \Delta_2 \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline A_{21} & I \end{array}\right)\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & I \end{array}\right)=\left(\begin{array}{c|c} I & B_1A_{12}\Delta_2 \\ \hline 0 & \Delta_2 \end{array}\right)\left(\begin{array}{c|c} B_1 & 0 \\ \hline A_{21}B_1 & I \end{array}\right) \tag{11}$$

In this vision of linkages, each region may be considered to exhibit a self-influence effect (through the standard Leontief influence) and through a push or pull relationship with the other region. Through matrix multiplication, the following Miyazawa formula may be obtained:

$$(I-A)^{-1}=\left(\begin{array}{c|c} B_1+B_1A_{12}\Delta_2A_{21}B_1 & B_1A_{21}\Delta_2 \\ \hline \Delta_2A_{12}B_1 & \Delta_2 \end{array}\right) \tag{12}$$

Equation (12) was known in the Frobenius/Schur school at the beginning of this century (for a historical review, see Henderson and Searle, 1981). The method used here is a variant of the well-known block form associated with the Gauss-Fourier-Jordan elimination method (see Gantmacher, 1959).

The multiplicative decomposition (11) presents two important features of regional synergetic interactions. First, each region is featured with a separate block-matrix regional multiplier of identical form and secondly, an hierarchy of interactions are revealed through the regional subsystems. In this case, for example, the block-matrix of the second region multiplier depends on the influence of the first region on the second region. Obviously, the "order" of the regions is important; if the second region is placed at the top of the hierarchy:

$$A=\left(\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & A_{22} \end{array}\right)+\left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & 0 \end{array}\right)=A_1'+A_2' \tag{13}$$

then:

$$G_1'=(I-A_1')^{-1}=\left(\begin{array}{c|c} I & -A_{12} \\ \hline 0 & I-A_{22} \end{array}\right)^{-1}=\left(\begin{array}{c|c} I & A_{12}B_2 \\ \hline 0 & B_2 \end{array}\right)=\left(\begin{array}{c|c} I & A_{12} \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & B_2 \end{array}\right) \tag{14}$$

where $B_2=(I-A_{22})^{-1}$ is the internal matrix multiplier for the second region.

Further,

$$G_2'=(I-G_1'A_2')^{-1}=\left(\begin{array}{c|c} I-A_{11}-A_{12}B_2A_{21} & 0 \\ \hline -B_2A_{21} & I \end{array}\right)^{-1}=\left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline B_2A_{21}\Delta_1 & I \end{array}\right)=\left(\begin{array}{c|c} I & 0 \\ \hline B_2A_{21} & I \end{array}\right)\left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline 0 & I \end{array}\right) \tag{15}$$

where

$$\Delta_1=(I-A_{11}-A_{12}B_2A_{21})^{-1} \tag{16}$$

is the external multiplier for the first region as it is influenced now by the second region.

Furthermore,

$$\begin{aligned}(I-A)^{-1}=G_2'G_1'&=\left(\begin{array}{c|c} I & 0 \\ \hline B_2A_{21} & I \end{array}\right)\left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & A_{12} \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & B_2 \end{array}\right)\\ &=\left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline B_2A_{21}\Delta_1 & I \end{array}\right)\left(\begin{array}{c|c} I & A_{12}B_2 \\ \hline 0 & B_2 \end{array}\right)=\left(\begin{array}{c|c} \Delta_1 & \Delta_1A_{12}B_2 \\ \hline B_2A_{21}\Delta_1 & B_2+B_2A_{21}\Delta_1A_{12}B_2 \end{array}\right)\end{aligned} \tag{17}$$

which reveals another version of the Miyazawa formula provided in (12). Essentially, (17) corresponds to the same set of regional sub-systems but with a transformation of the hierarchical arrangement of the regions.

A comparison of the components of the equations (12) and (17) yields the following equalities:

$$\begin{aligned}&\Delta_1=B_1+B_1A_{12}\Delta_2A_{21}B_1; \quad B_1A_{12}\Delta_2=\Delta_1A_{12}B_2\\ &\Delta_2=B_2+B_2A_{21}\Delta_1A_{12}B_2; \quad B_2A_{21}\Delta_1=\Delta_2A_{21}B_1\end{aligned} \tag{18}$$

Consider further, the following additive decomposition of the matrix of direct inputs:

$$A=\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)=\left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right)+\left(\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right)=A_1''+A_2'' \tag{19}$$

This decomposition represents the hierarchy of two sub-systems; the matrix, A_1'', corresponds to the intraregion (domestic) inputs in the two regions and the matrix, A_2'', captures the system of interregional inputs.

Consider the matrix:

$$G_1''=(I-A_1'')^{-1}=\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right) \tag{20}$$

and the matrix:

$$G_2''=(I-G_1''A_2'')^{-1}=\left(\begin{array}{c|c} I & -B_1A_{12} \\ \hline -B_2A_{21} & I \end{array}\right)^{-1} \tag{21}$$

The application of (12) and substituting A_{11} and A_{22} by zero matrices and A_{12}, A_{21} by B_1A_{21}, B_2A_{21} yields:

$$G_2''=\left(\begin{array}{c|c} I+B_1A_{12}\Delta_{22}B_2A_{21} & B_1A_{12}\Delta_{22} \\ \hline \Delta_{22}B_2A_{21} & \Delta_{22} \end{array}\right) \tag{22}$$

where:

$$\Delta_{22}=(I-B_2A_{21}B_1A_{12})^{-1} \tag{23}$$

may be interpreted as the Miyazawa external matrix multiplier for the second region.

Therefore:

$$(I-A)^{-1}=G_2''G_1''=\left(\begin{array}{c|c} B_1+B_1A_{12}\Delta_{22}B_2A_{21}B_1 & B_1A_{12}\Delta_{22}B_2 \\ \hline \Delta_{22}B_2A_{21}B_1 & \Delta_{22}B_2 \end{array}\right) \tag{24}$$

This presentation actually reflects the following hierarchy of three sub-systems, corresponding to the additive decomposition:

$$A=\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)=\left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right)+\left(\begin{array}{c|c} 0 & 0 \\ \hline A_{21} & 0 \end{array}\right)+\left(\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right) \tag{25}$$

A comparison of (24) with (12) provides:

$$\Delta_2=\Delta_{22}B_2 \tag{26}$$

which corresponds to Miyazawa's formulation. It may be interpreted as follows: the external matrix multiplier of the second region under the influence of inputs from the first region equals the internal multiplier of the second region premultiplied by the external matrix multiplier of the second region.

If (17) is substituted into (29) and, with further manipulation and substitution, another form of the matrix, G_2'', may be revealed:

$$G_2''=\left(\begin{array}{c|c} \Delta_{11} & \Delta_{11}B_1A_{12} \\ \hline B_2A_{21}\Delta_{11} & I+B_2A_{21}\Delta_{11}B_1A_{12} \end{array}\right) \tag{27}$$

where:

$$\Delta_{11}=(I-B_1A_{12}B_2A_{21})^{-1} \tag{28}$$

is the Miyazawa external multiplier of the first region.

From this form, a multiplicative decomposition may be obtained:

$$(I-A)^{-1}=\left(\begin{array}{c|c} \Delta_{11}B_1 & \Delta_{11}B_1A_{12}B_2 \\ \hline B_2A_{21}\Delta_{11}B_1 & B_2+B_2A_{21}\Delta_{11}B_1A_{12}B_2 \end{array}\right) \tag{29}$$

which corresponds to an hierarchy obtained from the decomposition:

$$A=\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)=\left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right)+\left(\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right)+\left(\begin{array}{c|c} 0 & 0 \\ \hline A_{21} & 0 \end{array}\right) \tag{30}$$

The comparison of (29) and (17) yields:

$$\Delta_1=\Delta_{11}B_1 \tag{31}$$

which may be interpreted as the external multipliers of the first region under the influence of the inputs from the second region and is equal to the internal multiplier of the first region premultiplied by the external multiplier for the first region.

Using (24) and (29) the following may be obtained:

$$(I-A)^{-1}=\left(\begin{array}{c|c}\Delta_{11}B_1 & \Delta_{11}B_1A_{12}B_2\\ \hline \Delta_{22}B_2A_{21}B_1 & \Delta_{22}B_2\end{array}\right)=\left(\begin{array}{c|c}\Delta_{11} & 0\\ \hline 0 & \Delta_{22}\end{array}\right)\left(\begin{array}{c|c}I & B_1A_{12}\\ \hline B_2A_{21} & I\end{array}\right)\left(\begin{array}{c|c}B_1 & 0\\ \hline 0 & B_2\end{array}\right) \tag{32}$$

which multiplicatively separates the Miyazawa internal and external, intraregional multipliers from the interregional effects. In terms of the system developed by Miller (1966, 1969), the first two matrices of (32) were combined and referred to as the interregional feedback effects. The advantage of (32) in this form is the separation of these feedback effects into external and push or pull effects.

Remark **1**: The parallel consideration of the following additive decomposition:

$$A=\left(\begin{array}{c|c}A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right)=\left(\begin{array}{c|c}A_{11} & A_{12}\\ \hline 0 & 0\end{array}\right)+\left(\begin{array}{c|c}0 & 0\\ \hline A_{21} & A_{22}\end{array}\right) \tag{33}$$

obviously leads to their associated "dual" formulae and interpretations.

Remark **2**: In this paper, the existence of different inverse matrices was postulated. Their existence is the consequence of the existence of the Leontief inverse, $(I-A)^{-1}$, (see Miyazawa, 1976).

III. *Generalizations for Multiregional Input-output Systems*

The previous section addressed the issues surrounding the use of the Miyazawa partition methods from the viewpoint of the multiplicative structure of the Leontief inverse. This structure reflects the division of the input-output system into a set of different economically and spatially meaningful sub-systems. However, it should be stressed that this multiplicative decomposition is essentially hierarchical in nature and thus depends on the way in which the economic sub-systems are ordered. As a result, different divisions and different hierarchies will map into different multiplicative decompositions.

As a direct result of this vision of the economy, the following general scheme of the multiplicative decomposition of the Leontief inverse can be proposed. Let the matrix of direct inputs, A, be additively decomposed into the sum of sub-matrices corresponding to the ordered set (hierarchy) of m sub-systems:

$$A=A_1+A_2+A_3+\ \dots\ +A_m \tag{34}$$

then:

$$I-A=I-A_1-A_2-A_3\dots-A_m$$

Consider:

$$G_1=(I-A_1)^{-1} \tag{35}$$

therefore:

$$G_1(I-A)=I-G_1A_2-G_1A_3-\ \dots\ -G_1A_m \tag{36}$$

Further define:

$$G_2=(I-G_1A_2)^{-1} \tag{37}$$

then:

$$G_2G_1(I-A)=I-G_2G_1A_3-\ \dots\ G_2G_1A_m \tag{38}$$

Continuing,

$$G_3=(I-G_2G_1A_3)^{-1} \tag{39}$$

which, after m steps the following equality holds:

$$G_mG_{m-1}\dots G_2G_1(I-A)=I \tag{40}$$

where:

$$G_k=(I-G_{k-1}G_{k-2}\dots G_2G_1A)^{-1} \quad k=1, 2, \dots, m \tag{41}$$

and, therefore, the multiplicative decomposition of the Leontief inverse is:

$$(I-A)^{-1}=G_mG_{m-1}\dots G_2G_1 \tag{42}$$

The Miyazawa partition, in which income generation is separated from output generation effects, is an example of such a presentation.

For a three-region scheme, the following pull-decomposition may be revealed:

$$A=\left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right)=\left(\begin{array}{c|c|c} A_{11} & 0 & 0 \\ \hline A_{21} & 0 & 0 \\ \hline A_{31} & 0 & 0 \end{array}\right)+\left(\begin{array}{c|c|c} 0 & A_{12} & 0 \\ \hline 0 & A_{22} & 0 \\ \hline 0 & A_{32} & 0 \end{array}\right)+\left(\begin{array}{c|c|c} 0 & 0 & A_{13} \\ \hline 0 & 0 & A_{23} \\ \hline 0 & 0 & A_{33} \end{array}\right)=A_1+A_2+A_3 \tag{43}$$

Drawing on the Gauss-Fourier-Jordan formalism, the inverse may be presented as:

$$(I-A)^{-1}=G_3G_2G_1=\left(\begin{array}{c|c|c} I & 0 & A_{13}'\Delta_{33} \\ \hline 0 & I & A_{23}'\Delta_{33} \\ \hline 0 & 0 & \Delta_{33} \end{array}\right)\left(\begin{array}{c|c|c} I & A_{12}'\Delta_{22} & 0 \\ \hline 0 & \Delta_{22} & 0 \\ \hline 0 & A_{32}'\Delta_{22} & I \end{array}\right)\left(\begin{array}{c|c|c} \Delta_{11} & 0 & 0 \\ \hline A_{21}\Delta_{11} & I & 0 \\ \hline A_{31}\Delta_{11} & 0 & I \end{array}\right) \tag{44}$$

where:

$$\begin{cases} \Delta_{11}=(I-A_{11})^{-1} \\ \Delta_{22}=(I-A_{22}-A_{21}\Delta_{11}A_{12})^{-1} \\ \Delta_{33}=(I-A_{33}-A_{21}\Delta_{11}A_{13}-[A_{32}+A_{21}\Delta_{11}A_{12}]\Delta_{22}[A_{23}+A_{21}\Delta_{11}A_{12}])^{-1} \end{cases} \tag{45}$$

and

$$\begin{cases} A_{12}'=\Delta_{11}A_{12}; \quad A_{32}'=A_{32}+A_{31}\Delta_{11}A_{12} \\ A_{23}'=A_{23}+A_{21}\Delta_{11}A_{12} \\ A_{13}'=\Delta_{11}A_{13}+\Delta_{11}A_{12}\Delta_{22}(A_{23}+A_{21}\Delta_{11}A_{12}) \end{cases} \tag{46}$$

Moreover,

$$G_1=\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline A_{21} & I & 0 \\ \hline A_{31} & 0 & I \end{array}\right)\left(\begin{array}{c|c|c} \Delta_{11} & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right) \tag{47}$$

$$G_2=\left(\begin{array}{c|c|c} I & A_{12}' & 0 \\ \hline 0 & I & 0 \\ \hline 0 & A_{32}' & I \end{array}\right)\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & \Delta_{22} & 0 \\ \hline 0 & 0 & I \end{array}\right) \tag{48}$$

$$G_3=\left(\begin{array}{c|c|c} I & 0 & A_{13}' \\ \hline 0 & I & A_{23}' \\ \hline 0 & 0 & I \end{array}\right)\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & \Delta_{33} \end{array}\right) \tag{49}$$

Equations (47) through (49) introduce the hierarchical sequences of regional matrix multipliers and reveal each region's structure through its own spatial multiplier, differentiating between the effects located within and between regions. Obviously, analogous formulae can be presented for any arbitrary number of regions.

IV. *Matrix Income Multipliers and General Demo-Economic Analysis*

Perhaps, the most important contribution of Miyazawa was associated with his analysis of the structure of income distribution (see Miyazawa, 1976 for the most complete exposition). The insights that he provided stimulated what may be referred to as an "onion-skin" approach to demographic-economic (hereafter, demo-economic) impact analysis (see Stone, 1981; Batey and Madden, 1981, 1983; Madden and Batey, 1980; Batey, 1985; Madden and Weeks, 1987; Sonis and Hewings, 1991 among others). All of these approaches attempted to link the demographic and economic parts of an economy in ways that would reveal the effects of changes in economic actions on income distribution, status in the labor force or migration behavior on the one hand and the effects of changes in consumption spending, employment status and so forth on economic activities. Miyazawa considered the following block matrix:

$$M=\left(\begin{array}{c|c} A & C \\ \hline V & 0 \end{array}\right) \tag{50}$$

where A is a block matrix of direct input coefficients, V a matrix of value-added ratios for some r-fold division of labor and non-labor categories and C is a corresponding matrix of consumption coefficients for the r-types of households.

Applying formula (11) to the Miyazawa matrix, M, yields:

$$(I-M)^{-1}=\left(\begin{array}{c|c} I & BC \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & K \end{array}\right)\left(\begin{array}{c|c} B & 0 \\ \hline VB & I \end{array}\right)=\left(\begin{array}{c|c} B(I+CKVB) & BCK \\ \hline KVB & K \end{array}\right) \tag{51}$$

where:

$B=(I-A)^{-1}$ is the Leontief inverse matrix

BC is a matrix of production induced by endogenous consumption

VB is a matrix of endogenous income earned from production

$L=VBC$ is a matrix of expenditures from endogenous income

$K=(I-L)^{-1}=(I-VBC)^{-1}$ is the Miyazawa interrelational multiplier or the generalized Keynesian multiplier

The application of (17) provides the following perspective:

$$(I-M)^{-1}=\left(\begin{array}{c|c} I & 0 \\ \hline V & I \end{array}\right)\left(\begin{array}{c|c} \Delta & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & C \\ \hline 0 & I \end{array}\right)=\left(\begin{array}{c|c} \Delta & \Delta C \\ \hline VB & I+VAC \end{array}\right) \tag{52}$$

where:

$$\Delta=(I-A-CV)^{-1}=B(I+CKVB) \tag{53}$$

is an enlarged Leontief inverse. Further, the following presentation of the generalized Keynesian multiplier may be revealed:

$$K=I+V\Delta C \tag{54}$$

and also the Miyazawa fundamental equations of income formation:

$$\begin{cases} V\Delta=KVB \\ \Delta C=BCK \end{cases} \tag{55}$$

Equations (52) and (53) provide the basis for extending the Miyazawa system to the case of several regions and several "onion skins," i.e., to the multi-regional demo-economic system. Consider first, a two-region scheme:

$$\Gamma=\begin{pmatrix} A_{11} & A_{12} & C_1 \\ A_{21} & A_{22} & C_2 \\ V_1 & V_2 & 0 \end{pmatrix} \tag{56}$$

where the matrix:

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

represents the direct inputs for the two regions and $V=(V_1,V_2)$; $C=(C_1,C_2)$ are matrices of value-added and consumption coefficients for both regions. Then, the enlarged Leontief inverse may be shown as:

$$\Delta=(I-A-CV)^{-1}=\left(\begin{array}{c|c} I-A_{11}-C_1V_1 & -A_{12}-C_1V_2 \\ \hline -A_{21}-C_2V_1 & I-A_{22}-C_2V_2 \end{array}\right)^{-1} \tag{57}$$

and the now familiar decomposition yields:

$$(I-\Gamma)^{-1}=\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline V_1 & V_2 & I \end{array}\right)\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\Delta} & 0 \\ \hline 0 & 0 & I \end{array}\right)\left(\begin{array}{c|c|c} I & 0 & C_1 \\ \hline 0 & I & C_2 \\ \hline 0 & 0 & I \end{array}\right) \tag{58}$$

Further, the enlarged Leontief inverse can be decomposed further through the use of (11), (17) and (24).

A general scheme for generalized demo-economic analysis can be elaborated. Consider the block matrix, M:

$$M=\begin{pmatrix} A & A_{12} & A_{13} & \cdots & A_{1m} \\ A_{21} & 0 & 0 & \cdots & 0 \\ A_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{m1} & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{59}$$

where, A, is the regional or multiregional matrix of direct interindustry coefficients while the block matrices:

$$V=\begin{pmatrix} A_{21} \\ A_{31} \\ \vdots \\ A_{m1} \end{pmatrix}; \qquad C=(A_{12} \quad A_{13} \quad \cdots \quad A_{1m}) \tag{60}$$

represent the demo-economic and the economic-demographic non-interindustry layers (or "skins"). From (50), the following result may be obtained:

$$(I-M)^{-1}=\begin{pmatrix} I & 0 & \cdots & 0 \\ A_{21} & I & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & 0 & \cdots & I \end{pmatrix}\begin{pmatrix} \Delta & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix}\begin{pmatrix} I & A_{12} & \cdots & A_{1m} \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots \\ 0 & 0 & \cdots & I \end{pmatrix}$$

$$=\left(\begin{array}{c|c|c|c} \Delta & \Delta A_{12} & \cdots & \Delta A_{1m} \\ \hline A_{21}\Delta & I+A_{21}\Delta A_{12} & \cdots & A_{21}\Delta\Delta_{1m} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline A_{m1}\Delta & A_{m1}\Delta A_{12} & \cdots & I+A_{m1}\Delta A_{1m} \end{array}\right) \tag{61}$$

where the enlarged Leontief inverse is:

$$\Delta=(I-A-A_{12}A_{21}-A_{13}A_{31}-\cdots-A_{1m}A_{m1})^{-1} \tag{62}$$

Increasing complexity can be built into these decomposed systems as the Miyazawa system is extended to a full set of social accounts (see Sonis and Hewings, 1991; Sonis, Hewings and Lee, 1993).

Conclusions

The major contribution that Miyazawa provided was the presentation of an analytical system of accounts that revealed the interdependencies between income and output generation processes. From the system of accounts that he provided, analysis of the income distribution effects of changes in sector outputs could be calculated in a straightforward fashion. His work has generated considerable interest and further developments in the field of demo-economic modeling. Empirical research has demonstrated on many occasions that the consumption-interindustry linkages are often the analytically most important elements in an economic system (see Sonis and Hewings, 1989, 1990, 1992; Sonis, Hewings and Lee, 1993). However, the specification of these linkages and methods for presentation have relied very heavily on the insights generated by Miyazawa's work. In this paper, some of his original ideas have been extended to reveal further insights through the application of a set of matrix decomposition techniques. In so doing, the policy analyst would now be in a much better position to trace the path of impacts and changes while maintaining the essential integrity of the original Miyazawa contribution. The extension to the multi-regional demo-economic framework represents an empirical challenge that has yet to be mounted although Madden (1985) has made some initial explorations.

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, USA AND BAR ILAN UNIVERSITY ISRAEL; AND UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, USA.

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